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THE CONSTRUCTION OF UNIVERSAL SURFACES IN EXTREMAL PROBLEMS OF DYNAMICS*

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One of the singular manifolds, often met in problems of optimal control and differential games, the universal hypersurface (the main line) /1/, is considered. The theory proposed here of that surface includes a certain Cauchy problem and the necessary condition of optimality of that surface. The latter enables us to reduce the construction of the surface to the Cauchy problem. The problem is formulated and sufficient conditions for its solvability are given. It is shown that the necessary condition of universal hypersurface optimality is the double smoothness of the optimum-result function on it. The sufficient condition for the existence in the small of a universal hypersurface is formulated. The paper is a continuation of investigations /2—4/ of the singular manifolds in extremal problems of dynamics, and the subject is related to /5—9/.

A universal manifold in the phase space of an external problem consists of (singular) optimal motions which from the neighbourhood of that manifold are incident on it an angle or are tangent to it. The universal hypersurface is the discontinuity surface of the positional /5/ optimal control /1/. The optimal-result function is continuous in the universal manifold neighbourhood and smooth on it, which follows from the geometry of trajectories. In the case of a hypersurface it appears that from the set of geometrically admissible surfaces the optimal one is that on which the optimal-result function is doubly smooth. This is one of the results of the investigation which to some extent solves the problem stated in /1/.

Using in the simplest version /1/ the inverse procedure, the indication of the presence in the problem of the universal surface is the appearance of a region free of characteristics. Hence, when constructing a universal hypersurface, the optimal-result function must be constructed simultaneously with the surface and on both sides of it. This defines the specific properties of the mathematical problem of constructing the surface. In connection with the discontinuity of optimal positional control /7/, it should be mentioned that on the singular manifold the function of minimum (or minimax in game problems), used to write the equation /1, 7/, is discontinuous.

Thus the singularity of the universal type of hypersurface is not related to the discontinuity of the optimal-result function and its first two derivatives. However, the procedure for constructing it is similar to that for constructing a weak discontinuity /3, 4/, i.e. the discontinuity manifold of the optimal-result function.

1. Statement of the Cauchy problem.

Let two scalar functions $u_0(x), u_1(x), x \in R^n$ of class $C^m(\Gamma_0)$ have on some hypersurface $\Gamma_1 \subset \Gamma_0 \subset R^n$ identical values of all partial derivatives with respect to the vector components $x = (x_1, \ldots, x_n)$ up to order m (where Γ_0 is the neighbourhood of some fixed point in R^n). We will assume that the pair of functions (u_0, u_1) smoothly join on Γ_1, and we will denote that inclusion by $(u_0, u_1) \in K^m(\Gamma_1)$, i.e. $K^m(\Gamma_1)$ is the set of all functions of class $C^m(\Gamma_0)$ that satisfy the conditions stated above.

Let a smooth manifold Γ_2, $\dim \Gamma_2 = n - 2$, $\Gamma_2 \subset \Gamma_0$ and doubly smooth function $v(x) \in C^2(\Gamma_0)$, $F_0(z)$, $F_1(z)$, $z = (x, p, u) \in R^{2n+1}$ be specified. We will assume the point $x^* \in \Gamma_2$ to be fixed and we will denote the neighbourhood by Γ_0.

Problem 1. To determine the smooth manifold Γ_1, $\dim \Gamma_1 = n - 1$ and the pair of functions $(u_0(x), u_1(x)) \in K^2(\Gamma_1)$ that satisfy the following conditions: 1) $\Gamma_2 \subset \Gamma_1 \subset \Gamma_0$; 2) $u_0(x) = u_1(x) = v(x)$, $x \in \Gamma_2$, and 3) $F_i(x, u_{ix}, u_i) = 0$, $x \in \Gamma_0$, $i = 0, 1$. The conditions of solvability of this problem and the procedure for obtaining its solution are given below in Sect.2.

2. Sufficient conditions of solvability.

When $x = \Gamma_2$, the gradient $p = u_{0x} = u_{1x}$ of the functions $u_i(x)$, $i = 0, 1$ that are sought in problem 1 satisfies the set of equations

$$(p - v_x, y_j) = 0, \quad j = 1, 2, \ldots, n - 2, \quad F_i(x, p, v) = 0 \tag{2.1}$$
$$i = 1, 2, \quad x \in \Gamma_2$$

where $y_j = y_j(x)$ is some basis of the space tangent to Γ_2 at the point x.

Theorem 1. Let the functions F_0, F_1 be triply smooth and the following conditions be satisfied: 1) the vector p^* is the solution of (2.1) at the point x^*; 2) when $x = x^*$, $p = p^*$, $v = v(x^*)$, the vectors $y_1, \ldots, y_{n-2}, F_{0p}, F_{1p}$ are linearly independent in R^n; 3) the

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inequalities $\{\{F_0, F_1\}, F_0\} \neq 0$. $\{F_1, \{F_0, F_1\}\} \neq 0$ hold at point (x^*, p^*, v^*), and 4) the equation $\{F_0, F_1\} = 0$ is satisfied, when $p = w(x), x \in \Gamma_2$. Then for each p^* a unique solution u_0, u_1, Γ_1 of Problem 1 exists. The surface Γ_1 consists of the x-components of the solutions of equations with the Cauchy conditions

$$
\begin{aligned}
& x^{\cdot} = \{\{F_0, F_1\}, F_0\} F_{1p} + \{F_1, \{F_0, F_1\}\} F_{0p} \\
& p^{\cdot} = -\{\{F_0, F_1\}, F_0\}(F_{1x} + pF_{1u}) - \{F_1, \{F_0, F_1\}\} (F_{0x} + \\
& \quad pF_{0u}). \quad u^{\cdot} = (p, x^{\cdot}) \\
& x(0) = x^{\circ}, \ p(0) = w(x^{\circ}), \ u(0) = v(x^{\circ}), \ x^{\circ} \in \Gamma_2 \\
& \{F_0, F_1\} = \{F_{0x} + pF_{0u}, F_{1p}\} - (F_{1x} + pF_{1u}, F_{0p})
\end{aligned}
\tag{2.2}
$$

where $p = w(x)$, $x \in \Gamma_2$ is a smooth solution of (2.1), and $\{F_0, F_1\}$ is the Jacobi bracket of the functions F_0, F_1 /10/.

Let us formulate statements that will be used to prove Theorem 1.

Lemma 1. Let the pair of functions $u, v \in C_2(\Gamma_0)$ smoothly join on the manifold Γ_1, $(u, v) \in K^1(\Gamma_1)$, where

$$\Gamma_1 \colon x_1 = 0 \tag{2.3}$$

For a doubly smooth union $(u, v) \in K^2(\Gamma_1)$ it is necessary and sufficient that at the point Γ_1 the following equations are satisfied:

$$u_{11} = v_{11}, \quad x \in \Gamma_1 \tag{2.4}$$

Here and below the element of the Hess matrix is denoted by $u_{ij} = \partial^2 u / \partial x_i \partial x_j$.

Proof. The necessity of equation (2.4) is obvious. By the conditions of the lemma $\partial u/\partial x_i = \partial v/\partial x_i$, $i = 1, \ldots, n$, when $x \in \Gamma_1$. We differentiate these equations with respect to $x_2, \ldots, x_n$, i.e. with respect to the tangents to the directions Γ_1. We obtain $u_{ij} = v_{ij}$, $i = 1, \ldots, n$, $j = 2, \ldots, n$. These equations together with (2.4) constitute the required doubly smooth union.

Lemma 2. Let the functions $u(x), v(x)$ satisfy, besides the conditions of Lemma 1, the equations: $F(x, u_x, u) = 0$, $G(x, v_x, v) = 0$, $x \in \Gamma_0$, $F(z), G(z) \in C^2$, $z = (x, p, u)$. Then at points of the surface Γ_1 the equation

$$\{F, G\} + F_{p_1} G_{p_1} (u_{11} - v_{11}) = 0, \quad x \in \Gamma_1 \tag{2.5}$$

holds.

We differentiate the identities $F(x, u_x, u) = 0$, $G(x, v_x, v) = 0$ with respect to x_i, $i = 1, \ldots, n$ and obtain

$$
\begin{aligned}
& F_{x_i} + \sum_{j=1}^{n} F_{p_j} u_{ji} + F_u p_i = 0 \quad (p = u_x) \\
& G_{x_i} + \sum_{j=1}^{n} G_{q_j} v_{ji} + G_u q_i = 0 \quad (q = v_x)
\end{aligned}
$$

Let us multiply the first equation termwise by G_{q_i} and the second by F_{p_i} and subtract the second from the first. Then adding the remainders obtained, related to the point $x \in \Gamma_1$, and taking into account that $q = p$, $u_{ij} = v_{ij}$ when $i + j > 1$, $x \in \Gamma_1$, we obtain (2.5).

Corollary. For every solution of Problem 1

$$\{F_0, F_1\} = 0 \quad \text{for} \quad p = u_{ix}(x), \quad x \in \Gamma_1$$

Lemma 3. Let the functions $u, v \in C^2(\Gamma_0)$ satisfy the conditions: 1) $F(x, u_x, u) = 0$, $G(x, v_x, v) = 0$, $x \in \Gamma_0$, $F, G \in C^2$; 2) $(u, v) \in K^1(\Gamma_1)$, where Γ_1 is a doubly smooth hypersurface, and 3) x are components of characteristics of the equations $F = 0$, $G = 0$, i.e. the vectors F_p, G_p do not touch the surface Γ_1. The condition $(u, v) \in K^2(\Gamma_1)$ is satisfied if and only if $\{F, G\} = 0$ for $p = u_x = v_x$, $u = v$, $x \in \Gamma_1$.

Proof. By the doubly smooth change of variables $x = \varphi(\xi)$, $\xi \in R^n$ the surface Γ_1 is reduced to the form $\xi_1 = 0$, i.e. (2.3). The change $x = \varphi(\xi)$ generates in space (x, p, u) a point contact transformation that maintains the Jacobi brackets /10/. Owing to the non-degeneracy of the Jacobian $\det \| \partial\varphi / \partial\xi \| \neq 0$, the condition of non-tangency to the surface Γ_1 is maintained in new variables. The statement of Lemma 3 then follows from (2.5).

Note that Lemma 3 also holds for the surface Γ_1 of class C^1. To prove this it is necessary to restate Lemmas 1 and 2 for the case of an arbitrary smooth surface Γ_1. Preference was given above to a particular form of (2.3) because of the simplicity of the calculations.

Proof of Theorem 1. The existence of a smooth solution $p = w(x)$, $x \in \Gamma_2$ of system (2.1) is ensured by condition 2) and the theorem on the implicit function.

Let us establish the existence of a solution of Problem 1. We set $F_{-1}(z) \equiv \{F_0, F_1\}$ and consider the Hamiltonian $H(z) = \lambda_1 F_1 + \lambda_0 F_0 + \lambda_{-1} F_{-1}$ when $\lambda_1 = \{\{F_0, F_1\}, F_0\}$, $\lambda_0 = \{F_1, \{F_0, F_1\}\}$, $\lambda_{-1} = F_{-1}$

/2, 4/.

Equations (2.2) represent the characteristic system

$$x^{\cdot} = H_p, \quad p^{\cdot} = -H_x - pH_u, \quad u^{\cdot} = (p, H_p)$$

consider on its own invariant manifold

$$W = \{z \in R^{2n+1}: F_0(z) = 0, \ F_1(z) = 0, \ F_{-1}(z) = 0\}$$

System (2.2) satisfies the standard conditions for the existence and uniqueness of a solution and its smooth dependence on the initial conditions /11, 12/.

The transversality of the vector $H_p(z^*)$ to the manifold Γ_2 is ensured by conditions 1) and 2) of the theorem. Hence the x components of the solution of system (2.2) define the smooth manifold $\Gamma_1 \supset \Gamma_2$. On Γ_1 we have the specified Cauchy conditions for the two equations $F_i = 0, i = 0, 1$. By virtue of condition (2) and 3) the vectors F_{0p}, F_{1p} prove to be transversal to the surface Γ_1, and hence the solution $u_i(x) \in C^2(\Gamma_0)$ of these equations exists /12/. By construction we have $\{F_0, F_1\} = 0$ on Γ_1, and hence the set u_0, u_1, Γ_1 is the solution of Problem 1. For any other solution of Problem 1 we have by virtue of Lemma 3 the condition $\{F_0, F_1\} = 0$ that is specified on Γ_1.

Condition 3) implies that the gradient g_x of the function $g(x) = \{F_0, F_1\}$ is non-zero, when $p = u_{ix}(x)$. Direct calculation shows that the vector $x^{\cdot}$ of the form (2.2) touches Γ_2, i.e. $(g_x, x^{\cdot}) = 0, x \in \Gamma_1$, and the derivative $p^{\cdot}$ of the vector $p(x)$ in the direction of $x^{\cdot}$ is defined by formula (2.2). System (2.2) satisfies the standard conditions for the existence and uniqueness of the solution whence follows the uniqueness of Γ_1. The functions $u_0(x), u_1(x)$ are also unique solutions of the respective Cauchy problems. The theorem is proved.

Note that instead of the vector p^* in condition 1) of Theorem 1, we may consider the vector field $p = w(x)$ and the function $v(x)$ on the manifold Γ_2 of class C^1, made compatible by (2.1), as specified in advance, as done in the theory of equations in first-order partial derivatives /12/.

3. Construction of the universal manifold. We shall use the results of Sects.1 and 2 for the local construction of a singular universal surface in the problem of synthesis for a dynamically controlled system of the form

$$x^{\cdot} = f(x, w); \quad w \in W; \quad x, f \in R^n \tag{3.1}$$

where $W \subset R^m$ is a closed bounded set. We shall consider the problem of otpimal high speed action from an arbitrary initial point $x \in M$ onto the terminal manifold $M \in R^n$.

Other problems of optimal control may be reduced to this problem by standard methods /1, 5/. Note also that the universal surface considered below occurs not only in problems of optimal control but, also, in the theory of differential games. In the latter case the system of the form (3.1) may be obtained after substituting into the equations of motion of a game problem, of the continuous positional control of one of the players /7/.

Let us write the assumptions and constraints that apply to the problem of high-speed action for system (3.1), which we shall limit to the open neighbourhood $\Gamma_0 \subset R^n$ of some fixed point $x^* \in R^n$. Let there be in Γ_0 an $(n-1)$-dimensional smooth universal surface Γ_1 to which the optimal trajectories approach from both sides without touching it. The surface Γ_1 divides the neighbourhood Γ_0, into two open regions $D_i : \Gamma_0 = D_0 + \Gamma_1 + D_1$. Instead of indicating the regions by D_i or Γ_0, we shall denote these regions by the expressions "the i-th side of surface" or "the neighbourhood of Γ_1." We set $x^* \in \Gamma_1$.

We denote the optimal-result function on the i-th side of Γ_1, $i = 0, 1$ by $u_i(x), x \in D_i$. By assumption $u_i(x) \in C^2(D_i)$ the function $u_0(x)$ and $u_1(x)$ smoothly join on the surface Γ_1 /8/. In regions D_i the functions $u_i(x)$ satisfy the basic equation /1, 5, 7/

$$F_i(x, u_{ix}(x)) = \min_{w \in W}(u_{ix}, f(x, w)) + 1 = (u_{ix}(x), f^i(x)) + 1 = 0, \quad x \in D_i \tag{3.2}$$

Formula (3.2) is written on the assumption (taken as satisfied) that the set $f(x, W)$ is convex. The functions $f^i(x)$ are obtained by substituting into the right side of (3.1) the extremal value of $w_i(x) = \psi_i(x, u_{ix}(x))$, which provides a minimum to (3.2). We assume that $w_i(x), x \in D_i$ are the unique points of minimum, and the quantities $w = \psi_i(x, p)$ provide the minimum of $\min_w(p, f(x, w)), w \in W$, when $p = u_{ix}$. We assume that the function $\psi_i(x, p)$ are determined in the neighbourhood of the point $(x^*, p^*) \in R^{2n}$, $\psi_i \in W$, and the functions

$$F_i(x, p) = (p, f(x, \psi_i(x, p))) + 1 \tag{3.3}$$

are doubly smooth in the neighbourhood of the point (x^*, p^*), where $p^* = u_{0x}(x^*) = u_{1x}(x^*)$. Note that generally the quantities $\psi_i(x, p)$ have the sense of an extremal vector not for all points (x, p) in the neighbourhood. This occurs only for (x, p), where $p = u_{ix}, x \in D_i$.

The constraints on the dynamic system have therefore been defined in terms of the functions F_i, ψ_i. Their properties enable us to supplement the definition of the functions $u_i(x)$ in the whole neighbourhood Γ_0 as doubly smooth equations $F_i(x, u_{ix}) = 0, i = 0,1$, with the Cauchy conditions on Γ_1, since for the universal surface without touching, the classical sufficient conditions for the existence and uniqueness of solutions are satisfied /12/. Then in Γ_0 two smooth fields $f^i(x) = F_{ip}(x, u_{ix}(x))$, $i = 0, 1$ are defined, which for $x \in D_i$ have the sense of the optimal velocity field in the extremal problem considered here.

The optimal phase velocity on the surface Γ_1 is given by the equation

$$x^{\cdot} = \mu(x) f^0(x) + (1 - \mu(x)) f^1(x), \quad 0 < \mu < 1 \tag{3.4}$$

where the scalar $\mu(x)$ is obtained from the condition that the vector (3.4) touches the surface Γ_1; the dependence $\mu = \mu(x)$, $x \in \Gamma_1$ is smooth. The inequalities in (3.4) follow from the assumptions about the geometry of optimal trajectories in the neighbourhood of Γ_1. From (3.2) and the equation $u_{0x} = u_{1x}$ when $x \in \Gamma_1$ it follows that the derivative functions u_{ix} in the direction (3.4) are equal to -1, i.e. the integral lines of the field (3.4) are optimal (singular) motions.

Since the vector diagram $f(x, W)$ is convex, a vector $w^*(x) \in W$, is obtained such that right side of (3.4) equals $f(x, w^*(x))$, $x \in \Gamma_1$. The phase velocity (3.4) corresponds to the form of sliding mode in which the velocities $f^0(x)$ and $f^1(x)$ are used with weighting factors $\mu(x)$ and $1 - \mu(x)$. The traditional procedure in investing singular motions consists of finding a singular control as a time function /6/. The motion over a singular manifold is constructed here in the form of a sliding mode, using the optimal controls in regions D_i. The control parameters on the surface Γ_1 are not considered here.

A singular optimal control may be re-established, after the solution of the problem, as the resulting control of the sliding mode.

The basic result, the necessary condition of optimality that allows the use of constructions developed in Sects.1 and 2, is formulated below for a doubly smooth surface Γ_1, which simplifies the proof. This result may also be established at the cost of some complication, for a smooth surface.

Lemma 4. Let Γ_1 be a universal surface of class C^2 for which the conditions 1) $0 < \mu(x) < 1$, $x \in \Gamma_1$ and 2) $F_i, u_i \in C^2$, $i = 0, 1$ are satisfied. Then u_0, u_1 twice smoothly join on Γ_1, i.e. $(u_0, u_1) \in K^2(\Gamma_1)$.

Proof. By a doubly smooth change of variables, which maintains the smoothness of the functions $u_i(x)$, the surface Γ_1 can be reduced to the form (2.2). Hence, without loss of generality, we assume that Γ_1 is defined by the equation $x_1 = 0$. By virtue of Lemma 1 it is sufficient to show that

$$\partial^2 u_0/\partial x_1^2 = \partial^2 u_1/\partial x_1^2, \quad x \in \Gamma_1 \tag{3.5}$$

The smooth function $\mu(x)$ is defined only on Γ_1. This definition is smoothly extended over the whole of Γ_0, using the equation $\mu(x_1, x_2, \ldots, x_n) = \mu(0, x_2, \ldots, x_n)$. Consider the smooth field $f^*(x) = \mu(x) f^0(x) + (1 - \mu(x)) f^1(x)$ and the derivatives of the function $u_i(x)$ in the direction of the field $h_i(x) = (u_{ix}(x), f^*(x))$, $x \in \Gamma_0$. By construction $h_i(x) = -1$ when $x \in \Gamma_1$. Using the equations obtained by differentiation of the identity (Bellman's equations) $(u_{ix}(x), f^i(x)) + 1 = 0$, we can derive for the partial derivatives $\partial h_i/\partial x_1$ for $x \in \Gamma_1$ the formulas

$$\frac{\partial h_i(x)}{\partial x_1} = (-1)^i \mu_i f_1^{1-i} \left(\frac{\partial^2 u_0}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_1^2}\right), \quad x \in \Gamma_1, \quad i = 0,1 \tag{3.6}$$

where $\mu_0 = \mu$, $\mu_1 = 1 - \mu$ and f_1^j is the first component of the vector $f^j(x)$. Note that by virtue of assumptions about the geometry of the trajectories, f_1^0 and f_1^1 are non-zero and have different signs. If we now assume that (3.5) does not apply, then one of the inequalities $\partial h_1/\partial x_1 < 0$, $\partial h_0/\partial x_1 > 0$, $x_* \in \Gamma_0$ follows from (3.6). This means that at the point $x_* \in \Gamma_0$ which is fairly close to Γ_1, the inequality $h_i(x_*) < 1$ is satisfied for one of the indices $i = 0, 1$ which contradicts the condition (3.2).

Thus, if in the problem of high speed action for system (3.1) a locally smooth universal surface Γ_1 exists for which $u_i(x)$, $F_i(x$, and $p) \in C^2$, then $(u_0, u_1) \in K^2(\Gamma_1)$. In other words, the set u_0, u_1, Γ_1 is the solution of Problem 1 for suitable Cauchy data. The second part of the proof of Theorem 1 implies that the vector $x^{\cdot}$ of the form (2.2), constructed from data of the problem u_i, F_i touches the surface Γ_1. Moreover, it proves to be collinear with the optimal phase velocity (3.4) on Γ_1.

Indeed, the vectors $x^{\cdot}$ in (2.2) and (3.4) emanate from the convex envelope of the vectors $F_{0p} = f^0(x)$ and $F_{1p} = f^1(x)$, and are tangent to Γ_1. Hence the two vectors $x^{\cdot}$ are collinear and the coefficients of F_{ip} are proportional, i.e.

$$\mu(1 - \mu)^{-1} = \{F_1, \{F_0, F_1\}\}\{\{F_0, F_1\}, F_0\}^{-1} \tag{3.7}$$

In (3.4) the dot on x denotes differentiation with respect to time and in (2.2) with respect to some ancillary parameter which differs by a scalar multiplier. The first equation of system (2.2) determines a sliding mode, if by a change of the differentiation parameter one makes the sum of the coefficients of F_{0p} and F_{1p} equal to unity. It follows from (3.7) that in the case of a universal surface the signs on the brackets $\{\{F_0, F_1\}, F_0\}$ and $\{F_1, \{F_0, F_1\}\}$ are the same. This and the equation $\{F_0, F_1\} = 0$, $x \in \Gamma_2$ are the necessary conditions for the presence of a universal surface. Both these conditions obtained by calculations on Γ_2 may be used for a preliminary check of the existence of a given type of surface.

Let us sum up the reasoning presented above in the form of the necessary conditions given below.

Lemma 5. Suppose that in the problem of high speed action of system (3.1) a smooth universal surface Γ_1 exists in the small, for which $u_i(x)$, $F_i(x, p) \in C^2$. Then the quantities x and $p = u_{ix}$ in singular motions on Γ_1 satisfy system (2.2) apart from the parameter of differentiation.

We emphasize that in the control problem the functions F_i are independent of u_i, which to some extent simplifies system (2.2).

Amplifying the smoothness requirements for the functions $F_i(x, p)$, we can formulate, with suitable initial conditions, the sufficient conditions for the unversal surface to exist in the small.

Suppose that in the phase space of system (3.1) the point x^* belong to some smooth hypersurface M which is divided by an $(n-2)$-dimensional manifold Γ_2 $x^* \in \Gamma_2$ into two semi-surfaces M_0, M_1 so, that $M = M_0 + \Gamma_2 + M_1$. Let the values $v(x)$ of the optimal high-speed action time $u(x)$ and the smooth field of its gradient $u_x = r(x)$, $x \in M$ be such that $F_i(x, r(x)) = 0$, $x \in M_i$, $i = 0, 1$. The quantities v, r can be obtained on M as the result of incomplete solution of the problem of the synthesis of the high-speed action for system (3.1). The surface M may, in particular, be a section of the terminal manifold, and v, r be the initial data for retrograde constructions. Let $l = l(x)$ be the normal to the surface M $x \in M$ oriented in the direction of the retrograde motion, i.e. $(F_{ip}, l) < 0$ when $p = r(x)$, $x \in M_i$, $i = 0, 1$. We denote by Γ_0^- the semineighbourhood of the point x^*, cut off by the surface M in the direction of the normal $l(x^*)$. We recall that the formulas $F_i(x, p)$ are defined in (3.3) above.

Theorem 2. Suppose that for system (3.1) the functions (3.3) are thrice smooth, $F_i(x, p) \in C^3$, $i = 0, 1$, and on the surface M for $p = r(x)$ the following conditions are satisfied:

1) $\{F_0, F_1\} = 0$, $x \in \Gamma_2$, $\sigma(-1)^i \{F_0, F_1\} < 0$, $x \in M_i$, $i = 0,1$;

2) $\{\{F_0, F_1\}, F_0\}\{F_1, \{F_0, F_1\}\} > 0$, $x \in \Gamma_2$, $\sigma = \operatorname{sgn}\{\{F_0, F_1\}, F_0\}$;

3) $(F_{ip}, l) < 0$, $F_i(x, p) = 0$, $w_i(x) = \psi_i(x, r(x))$, $x \in M_i$, $i = 0, 1$.

Then in region Γ_0^- smooth functions of the optimal result $u(x)$, $u(x) = v(x)$ exist when $x \in M$, and a universal hypersurface Γ_1 with edge Γ_2.

The scheme of proof. First, let us explain the conditions of the theorem. Condition 1) means that $\{F_0, F_1\}$ on M has different signs on each side of Γ_2. The double Poisson brackets in condition 2) must have the sign on Γ_2 that is consistent with the sign of $\{F_0, F_1\}$ on M_i. Conditions 1) and 2) ensure the characteristic divergence of optimal trajectories emerging in the reverse direction from points of M. As the result, a wedge-shaped region that arises on Γ_2 remains free of trajectories emerging (in the reverse direction) from the universal surface.

The surface Γ_1, whose existence is ensured by Theorem 1, divides the semicircle Γ_0^+ into two parts (by virtue of conditions 2 and 3). On the i-th side of Γ_1, in which the semisurface M_i lies, we shall consider the Cauchy problem of the equation $F_i(x, p) = 0$ with boundary condition on the piecewise smooth surface $M_i + \Gamma_2 + \Gamma_1$, $i = 0, 1$. Solutions $u_i(x)$ of these problems may be constructed using the method of characteristics emerging from Γ_1, Γ_2 and M_i. The possibility of a union of functions defined by the characteristics emerging from Γ_1 and M_i, and also the required direction of motion along them (reaching Γ_1 and M_i in the right time by the optimal motion) is ensured by conditions 1)-3). Thus a smooth function $u(x)$ equal to $u_i(x)$ on the i-th side of Γ_1 that satisfies Bellman's equation (3.2) with the boundary condition $u(x) = v(x)$, $x \in M$ is constructed. As in the proof of Theorem 3.15 in /13/, we establish that the function $u(x)$, $x \in \Gamma_0^-$ is equal to the time of optimal high-speed action from point x onto the set M.

In concluding, we point out that the question of double smoothness $u(x)$ depends on the smoothness of the union of the functions defined by sections of the boundaries of Γ_1 and M_i. However, here it is sufficient to have $u(x) \in C^1$.

4. Examples. In a number of problems of the theory of optimal control and of differential games the sufficient conditions of Theorem 2 are satisfied, particularly problems such as the maximization of the lift height of a probing rocket /5/, of the problem of the "killer driver", and the "two-car game" /1/. The dynamics of the latter are defined by the relations

$$x_1' = -x_2u - \nu\sin x_3, \quad x_2' = x_1u - 1 - \nu\cos x_3, \quad \nu < 1$$
$$x_3' = -u + \nu\gamma v, \quad |u| \leqslant 1, \quad |v| \leqslant 1$$

Consider a universal surface in whose neighbourhood $v = 1$. The Bellman-Isaacs equation of the initial problem has the form /1/

$$F(x, p) = \min_u \max_v H_1 + \nu\gamma p_3 v - Au = 0$$
$$H_1(x, p) = p_1\nu\sin x_3 - p_2\nu\cos x_3 - p_2 - 1$$
$$A(x, p) = -p_1x_2 + p_2x_1 - p_3$$

Minimizng this with respect to u and setting $v = 1$, we obtain (see (3.2))

$$F_0(x, p) = H - A = 0, \quad A > 0; \quad H = H_1 + \nu\gamma p_3 \tag{4.1}$$
$$F_1(x, p) = H - A = 0, \quad A < 0$$

The respective control $u(x, p) = -\operatorname{sgn} A$ in the notation of Sect.3 is $\psi_0(x, p) \equiv -1, \psi_1(x, p) \equiv 1$.

After parametrization, the terminal set M /1/ (it is a cylinder) is defined by the formulas $x_1 = l\sin s_1, x_2 = l\cos s_1, x_3 = s_2, -\pi < s_1, s_2 < \pi$. The transversality conditions yield

$$p_1 = \frac{\sin s_1}{\cos s_1 - \nu\cos(s_2 - s_1)}, \quad p_2 = \frac{\cos s_1}{\cos s_1 - \nu\cos(s_2 - s_1)}, \quad p_3 = 0 \tag{4.2}$$

According to the condition of Theorem 2 we must have on Γ_2 $F_0(x, p) = F_1(x, p) = \{F_0, F_1\} = 0$. But $F_0(x, p) \equiv F_1(x, p) \equiv 0$ on M, hence the condition $\{F_0, F_1\} \equiv 2\{H, A\} = -2p_1 = 0$ separates on M the line Γ_2^* whose parametric representation according to (4.2) is either Γ_2^*: $s_1 = 0$ or $x_1 = 0, x_2 = l$. We have on M $A \equiv p_3 \equiv 0$. However, using reasoning similar to that in /1/ when constructing the barrier in this game, it is possible to show that the line Γ_2 which belongs to Γ_2^* and corresponds to $v = 1$ is separated out by the relations $-\pi < s_2 < 0, s_1 = 0$. The neighbourhood Γ_0 of the point $x^* \in \Gamma_2$ defines on M two sets M_0 and M_1 separated by the line Γ_2 on which $s_1 < 0$ and $s_2 > 0$ are respectively satisfied, and the optimal control $u(x) = \psi_i(x, p(x)) = \operatorname{sgn} s_1$ is satisfied on M.

By direct calculation we obtain $\{\{F_0, F_1\}, F_0\} = \{F_1, \{F_0, F_1\}\} = -2p_2 < 0$, when $-\pi/2 < s_1 < \pi/2$. The sufficient conditions of Theorem 2 for the existence of a universal surface are, thus, satisfied in the neighbourhood of Γ_2.

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